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On Carleman and Knopp's Inequalities

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A sharpened version of Carleman's inequality is proved. This result unifies and generalizes some recent results of this type. Also the "ordinary" sum that serves as the upper bound is replaced by the corresponding Cesaro sum. Moreover, a Carleman-type inequality with a more general measure is proved and this result may also be seen as a generalization of a continuous variant of Carleman's inequality, which is usually referred to as Knopp's inequality. A new elementary proof of (Carleman–)Knopp's inequality and a new inequality of Hardy–Knopp type is pointed out.

Key Words: inequalities; Carleman's inequality; Knopp's inequality; geometric means; historical remarks.

1. INTRODUCTION

Carleman's inequality appeared in paper [5] on quasi-analytic functions. In that paper, Carleman gave necessary and sufficient conditions for functions not to be quasi-analytic. As a lemma (stated as a theorem) for one

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of the implications, Carleman proved that we, in fact, have

$$\sum_{k=1}^{\infty} \sqrt[k]{a_1 a_2 \cdots a_k} < e \sum_{k=1}^{\infty} a_k, \tag{1.1}$$

if $(a_k)_{k=1}^{\infty}$ is a sequence of real positive numbers and the sum on the right-hand side is convergent. The constant e is sharp.

Since Carleman published his results inequality (1.1) has been discussed, applied and generalized by several authors. Here, we just mention the following, all of which to some extent have guided us in our investigation: Hardy [8, 9] G. Pólya (see [4, p. 156]); Knopp [15] (see also [4, p. 487]); Carleson [6]; Redheffer [22]; Cochran and Lee [7]; Heinig [11]; Henrici [12]; Love [16]; Bicheng and Debnath [3]; Alzer [1]; Bennett [2]; Ping and Guozheng [21] and Pecaric and Stolarsky [18]. Let us just mention that some applications to continued fractions are given in [12] and that further references and information can be found in the recent interesting review article [18].

In this paper, we shall also consider the continuous analogue of (1.1), namely the inequality

$$\int_0^{\infty} \exp\left(\frac{1}{x}\int_0^x \log f(t)\, dt\right) dx < e \int_0^{\infty} f(x)\, dx, \tag{1.2}$$

which usually is referred to as Knopp's inequality (cf. [15, 10, p. 250]), but note that Hardy claims that (1.2) is due to Pólya. Also, this inequality has been generalized in a number of ways and here we just mention the fairly recent papers [13, 14, 17, 19, 20] and the references given in these papers.

In this paper, we state, prove and discuss a refinement and generalization of (1.1) (see Theorem 2.1) which, in particular, unifies and generalizes some recent results in [1, 3, 21]. For the proof of Theorem 2.1 we also prove a crucial lemma of independent interest because it may be regarded as a new generalization of the arithmetic–geometric mean (A–G) inequality.

We also prove a new (Carleman–Knopp type) inequality (Theorem 3.1) with a more general measure involved so that this new inequality contains both (1.1) and (1.2).

In fact, it is easy to see that (1.2) implies (1.1) (cf. our Section 4). In Section 4 we also present a new proof of (1.2) and this idea makes it possible to state a new Hardy–Knopp inequality (see Theorem 4.1).

Conventions: In this paper $(a_k)_{k=1}^N$, $N \in \mathbf{Z}_+$, denotes a sequence of nonnegative numbers and $(a_k^*)_{k=1}^N$ denotes the nonincreasing rearrangement of $(a_k)_{k=1}^N$. It will be tacitly understood that we have rearranged a sequence all over again for different values of N.

2. A SHARPENING OF CARLEMAN'S INEQUALITY

We begin with proving a generalization of the following well-known refinement of the A–G inequality (see [4, p. 98]):

$$\frac{a_1 + a_2 + \cdots + a_N}{N} - N\sqrt{a_1 a_2 \cdots a_N} \geqslant \frac{1}{N}\left(\sqrt{a_{\max}} - \sqrt{a_{\min}}\right)^2, \quad (2.1)$$

$N \in \mathbf{Z}_+$, where $a_{\max} = \max_{1 \leqslant i \leqslant N} \{a_i\}$, $a_{\min} = \min_{1 \leqslant i \leqslant N} \{a_i\}$.

LEMMA 2.1. *Let x_i, $i = 1, 2, \ldots, N$, be positive real numbers. We have*

$$A - G = \frac{x_1 + x_2 + \cdots + x_N}{N} - N\sqrt{x_1 x_2 \cdots x_N} \geqslant \frac{1}{N} \sum_{k=1}^{[N/2]} \left(\sqrt{x^*_{N-k+1}} - \sqrt{x^*_k}\right)^2, \quad (2.2)$$

where $(x^*_k)_{k=1}^N$ is the nonincreasing rearrangement of $(x_k)_{k=1}^N$.

Proof. Suppose that N is odd; the case when N is even is similar or even simpler. By the A–G inequality, we have

$$G = N\sqrt{x_1 x_2 \cdots x_N} = \left(x^*_{[N/2]+1}\right)^{1/N} \left(\sqrt{x^*_1 x^*_N}\right)^{2/N} \cdots \left(\sqrt{x^*_{[N/2]} x^*_{[N/2]+2}}\right)^{2/N}$$

$$\leqslant \frac{1}{N} x^*_{[N/2]+1} + \frac{2}{N} \sum_{k=1}^{[N/2]} \sqrt{x^*_{N-k+1} x^*_k}$$

$$= \frac{1}{N} x^*_{[N/2]+1} + \frac{1}{N} \sum_{k=1}^{[N/2]} \left(x^*_{N-k+1} + x^*_k\right) - \frac{1}{N} \sum_{k=1}^{[N/2]} \left(\sqrt{x^*_{N-k+1}} - \sqrt{x^*_N}\right)^2$$

$$= A - \frac{1}{N} \sum_{k=1}^{[N/2]} \left(\sqrt{x^*_{N-k+1}} - \sqrt{x^*_k}\right)^2,$$

and the proof is complete. ∎

Remark 1. By estimating the sum on the right-hand side by the first term we obtain (2.1).

We now use Lemma 2.1 to prove our sharpening of Carleman's inequality.

THEOREM 2.1. *Let $(a_k)_{k=1}^\infty$ be a sequence of positive real numbers and let $x_i = i a_i (1 + \frac{1}{i})^i$, $i = 1, 2, \ldots$.*

Then with $G_k := k\sqrt{a_1 a_2 \cdots a_k}$ *and* $l_k := \sum_{i=1}^{[k/2]} \left(\sqrt{x^*_{k-i+1}} - \sqrt{x^*_i}\right)^2$ *we have*

$$\sum_{k=1}^{N} G_k + \sum_{k=1}^{N} \frac{l_k}{k(k+1)} \leqslant \sum_{k=1}^{N} \left(1 - \frac{k}{N+1}\right)\left(1 + \frac{1}{k}\right)^k a_k \tag{2.3}$$

for every $N \in \mathbf{Z}_+$.

Proof. We apply Lemma 2.1 with $x_i := ia_i$ and obtain

$$k\sqrt{k!}\left(\prod_1^k a_i\right)^{1/k} = \prod_{i=1}^{k} (ia_i)^{1/k} \leqslant \frac{1}{k}\sum_{i=1}^{k} ia_i - \frac{1}{k}\sum_{i=1}^{[k/2]}\left(\sqrt{x^*_{k-i+1}} - \sqrt{x^*_i}\right)^2.$$

Thus

$$\begin{aligned} &\sum_{k=1}^{N} \frac{1}{k+1} k\sqrt{k!}\left(\prod_1^k a_i\right)^{1/k} + \sum_{k=1}^{N} \frac{l_k}{k(k+1)} \\ &\leqslant \sum_{k=1}^{N} \frac{1}{k(k+1)} \sum_{i=1}^{k} ia_i = \sum_{k=1}^{N}\left(1 - \frac{k}{N+1}\right) a_k. \end{aligned} \tag{2.4}$$

By replacing a_k with $a_k(1 + \frac{1}{k})^k$, $k = 1, 2, \ldots, N$, in (2.4) this inequality coincides with (2.3), since, $(k+1)^k = k! \prod_{i=1}^{k}(1 + \frac{1}{i})^i$. ∎

Remark 2. By letting $N \to \infty$ and using the estimate $l_k \geqslant 0$, we obtain the classical Carleman inequality (1.1) for a convergent sum $\sum_1^\infty a_k$. It is then obvious that we obtain a strict inequality, since it is only when all numbers are equal we get equality in the A–G inequality, but then the right-hand side diverges.

Remark 3. Improvements with e replaced by $(1 + \frac{1}{k})^k$ in Carleman's inequality have been known since at least 1967, see e.g. [22] or [18, p. 53]. Moreover, the factor $1 - \frac{k}{N+1}$ in our formulation means that on the right-hand side the "usual" sum has been replaced by the Cesaro sum, i.e., the partial sums have been averaged arithmetically. This is of course strictly smaller than the ordinary sum because here the summands are nonnegative.

COROLLARY 2.1. *Let* $(a_k)_{k=1}^\infty$ *be a sequence of positive real numbers and let* $x_i = ia_i$, $i = 1, 2, \ldots$.

Then, with G_k *and* l_k *as in Theorem* 2.1, *we have*

$$\frac{1}{e}\sum_{k=1}^{N} G_k + \sum_{k=1}^{N} \frac{l_k}{k(k+1)} < \sum_{k=1}^{N}\left(1 - \frac{k}{N+1}\right) a_k, \tag{2.5}$$

for every $N \in \mathbf{Z}_+$.

Proof. Apply Theorem 2.1 with a_i replaced by $a_i(1+\frac{1}{i})^{-i}$ and use the estimate

$$\sqrt[k]{\prod_{i=1}^{k}\left(1+\frac{1}{i}\right)^{i}} < e$$

in (2.3) and the result follows. ∎

Remark 4. By letting $N \to \infty$ in (2.5) we obtain

$$\frac{1}{e}\sum_{k=1}^{\infty} G_k + \sum_{k=1}^{\infty}\frac{l_k}{k(k+1)} < \sum_{k=1}^{\infty} a_k,$$

when the right-hand side is convergent. This is a sharper statement than that of Alzer [1]. The Alzer result is obtained if l_k (which is a sum) is replaced by the first term of l_k.

COROLLARY 2.2. *Let $(a_k)_{k=1}^{\infty}$ be a sequence of positive real numbers and let $x_i = ia_i(1+\frac{1}{i})^i$, $i = 1, 2, \ldots$. Then, with G_k and l_k as in Theorem* 2.1, *we have*

$$\sum_{k=1}^{N} G_k + \sum_{k=1}^{N}\frac{l_k}{k(k+1)} < e\sum_{k=1}^{N}\left(1+\frac{1}{k+\frac{1}{5}}\right)^{-1/2}\left(1-\frac{k}{N+1}\right)a_k \tag{2.6}$$

for every $N \in \mathbf{Z}_+$.

Proof. By using Lemma 1 in [21] we have that

$$\left(1+\frac{1}{k}\right)^{k} < e\left(1+\frac{1}{k+\frac{1}{5}}\right)^{-1/2},$$

so (2.6) follows from (2.3). ∎

Remark 5. By letting $N \to \infty$ in (2.6) for a convergent right-hand side we obtain

$$\sum_{k=1}^{\infty} G_k + \sum_{k=1}^{\infty}\frac{l_k}{k(k+1)} < e\sum_{k=1}^{\infty}\left(1+\frac{1}{k+\frac{1}{5}}\right)^{-1/2} a_k. \tag{2.7}$$

This is a sharpened version of the inequality stated in [21, Theorem 1]. In fact, this inequality is obtained by just using the estimate $l_k \geqslant 0$ in (2.7).

Remark 6. By arguing as above we find that Theorem 2.1 also implies

$$\sum_{k=1}^{\infty} G_k + \sum_{k=1}^{\infty} \frac{l_k}{k(k+1)} < e \sum_{k=1}^{\infty} \left(1 - \frac{1}{2(k+1)}\right) a_k. \tag{2.8}$$

This is a sharpened version of the inequality stated in [3, Theorem 3.1]. Their result is obtained from (2.8) by replacing all l_k with 0.

3. A CARLEMAN–KNOPP INEQUALITY

In this section we prove an inequality which, in particular, generalizes and unifies the two inequalities (1.1) and (1.2).

We assume that $M(t)$ is a right-continuous and nondecreasing function on $[0, \infty)$. Moreover, let $g(t)$ be a continuous and increasing function on $(0, \infty)$ and let $G(x) = \int_0^x g(t)\, dt$. We define the function g^* by

$$g^*(M(x)) = \begin{cases} g(M(x)) & \text{if } M \text{ is continuous at } x, \\ \dfrac{G(M(x+)) - G(M(x-))}{M(x+) - M(x-)} & \text{elsewhere.} \end{cases}$$

In particular, we obviously have that

$$\int_0^x g^*(M(t))\, dM(t) = G(M(x)) \tag{3.1}$$

and

$$g^*(M(x)) \leqslant g(M(x)). \tag{3.2}$$

Our Carleman–Knopp inequality reads as

THEOREM 3.1. *Let* $M_*(t) := \exp(\log^*(M(t)))$. *Then, for any* $B \in \mathbf{R}_+$,

$$\int_0^B \exp\left\{\frac{1}{M(x)} \int_0^x \log f(t)\, dM(t)\right\} dM(x)$$

$$+ e \int_0^B \left(1 - \frac{M_*(x)}{M(x)}\right) f(x)\, dM(x) \leqslant e \int_0^B \left(1 - \frac{M_*(x)}{M(B)}\right) f(x)\, dM(x).$$

Proof. By using (3.1) with $g(t) = \log t$, Jensen's inequality and Fubini's theorem, we find that

$$\int_0^B \exp\left\{\frac{1}{M(x)}\int_0^x \log f(t)\,dM(t)\right\} dM(x)$$

$$= \int_0^B \exp\left\{\frac{1}{M(x)}\int_0^x [\log^* M(t) + \log f(t) - \log^* M(t)]\,dM(t)\right\} dM(x)$$

$$= \int_0^B \exp\left\{\frac{1}{M(x)}\int_0^x (\log^* M(t) + \log f(t))\,dM(t) - \frac{1}{M(x)}\int_0^x \log^* M(t)\,dM(t)\right\} dM(x)$$

$$= \int_0^B \frac{e}{M(x)}\left[\exp\left\{\frac{1}{M(x)}\int_0^x (\log^* M(t) + \log f(t))\,dM(t)\right\}\right] dM(x)$$

(here we use Jensen's inequality)

$$\leqslant \int_0^B \frac{e}{M(x)}\frac{1}{M(x)}\left(\int_0^x \exp(\log^* M(t)) f(t)\,dM(t)\right) dM(x)$$

$$= e\int_0^B M_*(t) f(t)\left(\int_t^B \frac{1}{(M(x))^2}\,dM(x)\right) dM(t)$$

$$= e\int_0^B M_*(t) f(t)\left(\frac{1}{M(t)} - \frac{1}{M(B)}\right) dM(t)$$

$$= e\int_0^B \left(\frac{M_*(t)}{M(t)} - 1\right) f(t)\,dM(t) + e\int_0^B \left(1 - \frac{M_*(t)}{M(B)}\right) f(t)\,dM(t),$$

which gives the desired inequality. ∎

COROLLARY 3.1. *Let $M(\infty) = \infty$. Then*

$$\int_0^\infty \exp\left\{\frac{1}{M(x)}\int_0^x \log f(t)\,dM(t)\right\} dM(x) + e\int_0^\infty \left(1 - \frac{M_*(x)}{M(x)}\right) f(x)\,dM(x) < e\int_0^\infty f(x)\,dM(x), \tag{3.3}$$

whenever the integral on the right-hand side converges.

Proof. Except for the strict inequality sign in the second row of (3.3) the proof follows by just letting $B \to \infty$ in Theorem 3.1. The strict inequality is obvious because in order to have equality in the (Jensen) inequality in the proof of Theorem 3.1 we must have $f(t) = \text{constant}$ a.e., but this is not possible when $B \to \infty$. ∎

Remark 7. By applying Corollary 3.1 with

$$M(x) = \begin{cases} 1/2, & 0 \leqslant x \leqslant 1, \\ k, & k \leqslant x < k+1, \ k = 1, 2, \dots, \end{cases}$$

we obtain (a slight generalization of) (1.1). Moreover, if $M(x) = x$, then (3.3) just coincides with (1.2).

We give a single example for the case $M(\infty) < \infty$.

EXAMPLE 3.1. Let $M(x) = 1 - e^{-x}$ in Theorem 3.1 and let $B \to \infty$. Then we obtain the inequality

$$\int_0^\infty \exp\left(\frac{e^x}{e^x - 1} \int_0^x e^{-t} \log f(t)\, dt\right) e^{-x}\, dx \leqslant e \int_0^\infty f(x) e^{-2x}\, dx.$$

4. CONCLUDING REMARKS AND RESULTS

Remark 8. It is easy to see that Knopp's inequality (1.2) implies Carleman's inequality (1.1). In fact, apply (1.2) with $f(x) = a_k$, $x \in [k-1, k)$, $k = 1, 2, \dots$. Then, by making some straightforward calculations and estimates, we see that (1.2) Carleman implies the inequality

$$\sum_{k=1}^{\infty} \sqrt[k]{a_1 a_2 \cdots a_k} < e \sum_{k=1}^{\infty} a_k. \tag{4.1}$$

The crucial estimate is

$$\int_k^{k+1} \exp\left(\frac{1}{x} \sum_{i=1}^{k} \log a_i + \frac{x-k}{x} \log a_{k+1}\right) dx$$
$$\geqslant \int_k^{k+1} \exp\left(\frac{1}{k+1} \sum_{i=1}^{k+1} \log a_i\right) dx = \left(\prod_{i=1}^{k+1} a_i\right)^{1/(k+1)},$$

which holds for each nonincreasing sequence and it is obviously sufficient to prove (1.1) or (4.1) for such sequences.

Remark 9. The original proof of Carleman was based on the Lagrange multiplier method (see [5]). Other proofs are based on *Hardy's inequality* (see [9, p. 156; 10]), or various formulations of *the A–G inequality* (see e.g.

[9, p. 77; 10, p. 249; 7, p. 24]), or *convexity* (see [6]). Still some other methods of proof are presented in [18]. Here, we shall present a new and in our opinion more elementary proof which also only depends on a convexity argument.

Proof of (1.2). First, we note that by replacing $f(t)$ with $f(t)/t$ in (1.2) we find that (1.2) can be rewritten in the equivalent—and in our opinion more natural—form

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \log f(t)\,dt\right)\frac{dx}{x} < \int_0^\infty f(x)\frac{dx}{x}. \tag{4.2}$$

In order to prove (4.2) we just use the fact that the function $f(u) = e^u$ is convex and apply Jensen and Fubini's inequalities to obtain

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \log f(t)\,dt\right)\frac{dx}{x} \leqslant \int_0^\infty \frac{1}{x^2}\left\{\int_0^x f(t)\,dt\right\}dx$$

$$= \int_0^\infty f(t)\left\{\int_t^\infty \frac{1}{x^2}\,dx\right\}dt = \int_0^\infty f(t)\frac{dt}{t}.$$

The strict inequality follows because in order to have equality in Jensen's inequality for almost all x it is necessary that $f(x)$ is constant almost everywhere, but this contradicts the assumption that

$$\int_0^\infty f(x)\frac{dx}{x} < \infty. \quad \blacksquare$$

Remark 10. Our proof of Theorem 2.1 and hence of (1.1) was based on the numbers x_i with $a_i = x_i/i$ while the proof above may be seen as based on the analogous fact that $f(x)$ is written in the form $g(x)/x$.

According to the proof above we find that the following *Hardy– Knopp-type inequality* holds:

THEOREM 4.1. *Let ϕ be a positive convex strictly increasing function on* $(0, \infty)$. *Then*

$$\int_0^\infty \phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x} \leqslant \int_0^\infty \phi(f(x))\frac{dx}{x}. \tag{4.3}$$

Proof. Using our proof of (4.2) above, we see that

$$\int_0^\infty \phi\left(\frac{1}{x}\int_0^x \phi^{-1}(f(t))\,dt\right)\frac{dx}{x} \leqslant \int_0^\infty f(x)\frac{dx}{x},$$

where ϕ^{-1} denotes the inverse of ϕ. Now, replace $f(x)$ by $\phi(f(x))$ and (4.3) follows immediately. ∎

Remark 11. By choosing $\phi(u) = e^u$ and $f(u) = \log g(u)$ we find that (4.3) implies (4.2) and by choosing $\phi(u) = u^p$ we find that (4.3) implies Hardy's inequality in the particular form

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p \frac{dx}{x} \leqslant \int_0^\infty f^p(x)\frac{dx}{x}, \qquad p \geqslant 1, \tag{4.4}$$

which for the case $p > 1$ (after some straightforward calculations) can be rewritten in the usual form

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(t)\,dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty g^p(x)\,dx, \qquad p > 1, \tag{4.5}$$

where $g(x) = f(x^{(p-1)/p})x^{-1/p}$. Note that Hardy's inequality written in form (4.4) in fact also holds for $p = 1$, but this has no meaning when it is written in form (4.5).

Remark 12 (On the Sharpness in (2.6)–(2.8)). Let $c^* = (8 - e^2)/(e^2 - 4) \approx 0.1802696$ and consider the sequence

$$b_k = \left(1 + \frac{1}{k}\right)^k \left(1 + \frac{1}{k + c^*}\right)^{1/2}.$$

We note that

$$b_1 = e \quad \text{and} \quad b_k < e, \text{ for } k = 2, 3, 4, \ldots \tag{4.6}$$

(because $b_2 < e$ and $\{b_k\}_2^\infty$ is strictly increasing and $b_k \to e$ as $k \to \infty$).

For this reason, we see from Theorem 2.1 that both (2.6) and (2.7) can be improved by replacing the factor $\left(1 + \frac{1}{k+\frac{1}{5}}\right)^{-1/2}$ with $\left(1 + \frac{1}{k+c^*}\right)^{-1/2}$ in these inequalities.

We also note that (4.6) does not hold if c^* is replaced by any smaller number, so with this technique inequalities (2.6) and (2.7) cannot be further improved.

Moreover, we note that

$$\left(1 + \frac{1}{k}\right)^k < e\left(1 - \frac{a}{k}\right) \tag{4.7}$$

for all k when $a \leqslant 1/2$, but not when $a > 1/2$. Also, we have by Taylor expansion

$$\left(1+\frac{1}{k}\right)^k = e\left(1-\frac{1}{2k}\right) + \mathcal{O}\left(\frac{1}{k^2}\right). \tag{4.8}$$

We can rewrite (4.8) into the form

$$\left(1+\frac{1}{k}\right)^k = e\left(1-\frac{1}{2(k+1)}\right) + \mathcal{O}\left(\frac{1}{k^2}\right),$$

which means that the corresponding estimate in (2.8) cannot be further improved only by using an estimate of form (4.7).

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REFERENCES

1. H. Alzer, A refinement of Carleman's inequality, *J. Approx. Theory* **95** (1998), 497–499.
2. G. Bennett, Inequalities complementary to Hardy, *Quart. J. Math. Oxford* **49** (1998), 395–432.
3. Y. Bicheng and L. Debnath, Some inequalities involving the constant e and an application to Carleman's inequality, *J. Math. Anal. Appl.* **223** (1998), 347–353.
4. P. S. Bullen, D. S. Mitronivic, and P. M. Vasic, "Means and Their Inequalities," D. Reidel Publication Company, Dordrecht, Boston, Lancaster, Tokyo, 1988.
5. T. Carleman, Sur les fonctions quasi-analytiques, "Comptes rendus du V^e Congres des Mathematiciens," pp. 181–196, Scandinaves, Helsingfors, 1922.
6. L. Carleson, A proof of an inequality of Carleman, *Proc. Amer. Math. Soc.* **5** (1954), 932–933.
7. J. A. Cochran and C. S. Lee, Inequalities related to Hardy's and Heinig's, *Math. Proc. Cambridge Philos. Soc.* **96** (1984), 1–7.
8. G. H. Hardy, Notes on some points in the integral calculus, LX, *Messenger of Math.* **54** (1925), 150–156.
9. G. H. Hardy, Prolegomena to a chapter on inequalities, *J. London Math. Soc.* **4** (1929), 61–78.
10. G. H. Hardy, J. E. Littlewood, and G. Pólya, "Inequalities," 2nd ed., Cambridge Univ. Press, Cambridge, UK, 1952 (1934).
11. H. P. Heinig, Some extensions of Hardy's inequality, *SIAM J. Math. Anal.* **6** (1975), 698–713.

12. P. Henrici, ''Applied and Computational Complex Analysis,'' Vol II, Wiley, New York, 1977.
13. P. Jain, L. E. Persson, and A. Wedestig, From Hardy to Carleman and general mean-type inequalities, *in* ''Function Spaces and Applications,'' (D. E. Edmunds *et al.* Eds.), pp. 117–130, Narosa Publishing House, New Delhi, 2000.
14. P. Jain, L. E. Persson, and A. Wedestig, Carleman–Knopp type inequalities via Hardy inequalities, *Math. Ineq. Appl.* **4** (2001), 343–355.
15. K. Knopp, Über Reihen mit positive Gliedern, *J. London Math. Soc.* **3** (1928), 205–211.
16. E. R. Love, Inequalities related to those of Cochran and Lee, *Math. Proc. Cambridge Philos. Soc.* **99** (1986), 395–408.
17. B. Opic and P. Gurka, Weighted inequalities for geometric means, *Proc. Amer. Math. Soc.* **120** (1994), 771–779.
18. J. Pecaric and K. B. Stolarsky, Carleman's inequality: history and new generalizations, *Aequationes Math.* **61** (2001), 49–62.
19. L. E Persson and V. Stepanov, Weighted integral inequalities with the geometric mean operator, *J. of Inequal & Appl.*, to appear.
20. L. Pick and B. Opic, On geometric mean operator, *J. Math. Anal. Appl.* **183** (1994), 652–662.
21. Y. Ping and S. Guozheng, A strengthened Carleman inequality, *J. Math. Anal. Appl.* **240** (1999), 290–293.
22. R. M. Redheffer, Recurrent inequalities, *Proc. London Math Soc.* **17** (1967), 633–699.